

Lifchitz tail and sausage asymptotics for stable processes in the Poissonian environment on the Sierpiński gasket

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Abstract

We obtain the Lifschitz tail asymptotics for the integrated density of states of the subordinate α -stable processes on the Sierpiński gasket \mathcal{G} , evolving among killing Poissonian obstacles. Simultaneously, we derive the large-time asymptotics for the volume of the α -stable sausage on the gasket.

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1 Introduction

The purpose of this paper is to obtain the Lifschitz tail for the integrated density of states of random Schrödinger operators based on fractional laplacians on the Sierpiński gasket perturbed by killing Poissonian obstacles.

The integrated density of states (IDS, for short) comes into play in the analysis of large-volume systems, when the properties of the spectra of the infinite-volume hamiltonians are difficult to capture. In classical cases – when the hamiltonian is based on the Laplace operator in \mathbb{R}^d – it has been thoroughly examined (see e.g. [5], [19] for a review). For Poissonian-type interaction, its existence and behaviour near zero have been analysed also in some nonclassical cases (hyperbolic space [22], Sierpiński gasket [16], general nested fractals [18]). All these papers were concerned with diffusion processes (whose generators are local operators).

For nonlocal operators, the results are not as abundant. In the classical case (i.e. that of Lévy processes on \mathbb{R}^d) the existence of the IDS in ergodic random environment and the asymptotics of related functionals was investigated in [15]. Recently, the existence of the IDS for subordinate Brownian motions on the Sierpiński gasket evolving in random Poissonian environment has been established in [11]. Now we will examine the behaviour of the IDS near zero for subordinate stable processes on the gasket and we will show that the Lifschitz tail is present in this case, with exponents reflecting the specific scaling of stable processes. We will also determine the asymptotics of the stable sausage on \mathcal{G} – the two are closely related. The methods we use rely on the enlargement of obstacles technique designed by Sznitman [21], but adapted to the nondiffusive setting.

More precisely, for $\alpha \in (0, 2)$, we consider the symmetric α -stable process on the unbounded Sierpiński gasket $\mathcal{G} \subseteq \mathbb{R}^2$, and an independent Poisson point process on \mathcal{G} with intensity $\nu\mu$, where μ is the Hausdorff measure on \mathcal{G} in dimension $d_f = \frac{\log 3}{\log 2}$ and $\nu > 0$ is given. The points of the Poisson process are centers of balls with radius a which we call obstacles. The stable process is killed after

entering one of the obstacles. Typically, the semigroup corresponding to this process is not be trace-class and so the spectrum of its generator may not be discrete. To get hold on some properties of the spectrum one considers the process in large balls $\mathcal{G}^{(M)}$ of diameter 2^M . The stable process is then killed when it comes to the obstacle set, or when it jumps out of the set $\mathcal{G}^{(M)}$. We are interested in the spectra of the generators of this processes, $\mathcal{L}^M = \mathcal{L}^{M,\omega}$. Now the semigroups become trace-class, so these spectra are pure point and consist of eigenvalues without nontrivial accumulation points. For each M we build the empirical measure based on these random sequences of eigenvalues and normalize them by dividing by the volume of the $\mathcal{G}^{(M)}$. These empirical random measures, denoted by $l(M, \omega)$, converge vaguely, when $M \rightarrow \infty$, to a deterministic measure l on $[0, \infty)$ which is by definition the integrated density of states. We will prove that the IDS fulfills the following property: there exist two constants: $C > 0$ and $D > 0$ such that

$$-C\nu \leq \liminf_{\lambda \rightarrow 0} \lambda^{d_f/(\alpha d_w)} \log l([0, \lambda]) \leq \limsup_{\lambda \rightarrow 0} \lambda^{d_f/(\alpha d_w)} \log l([0, \lambda]) \leq -D\nu, \quad (1.1)$$

where $d_w = \frac{\log 5}{\log 2}$ is the walk dimension of \mathcal{G} . It shows that the decay of l close to zero is exponential – faster than for the IDS of the nonrandom stable hamiltonian, which is only polynomial. This is the Lifschitz tail asymptotics, first discovered in 1965 by Lifschitz for disordered quantum systems, and subsequently rigorously proven to hold in various other models (see e.g. the reference list of [3]).

To get the desired result, we first derive the asymptotics for the Laplace transform of the measure l (denoted by L): there are two positive constants C_1, D_1 such that

$$-C_1\nu^{\frac{\alpha}{2}d_w/d_\alpha} \leq \liminf_{t \rightarrow \infty} \frac{\log L(t)}{t^{d_f/d_\alpha}} \leq \limsup_{t \rightarrow \infty} \frac{\log L(t)}{t^{d_f/d_\alpha}} \leq -D_1\nu^{\frac{\alpha}{2}d_w/d_\alpha},$$

where $d_\alpha = d_f + \alpha d_w/2$. To get (1.1), we employ a Tauberian theorem of exponential type from [10].

Simultaneously, we establish the asymptotics for the stable sausage on the gasket in large time: we prove that there exist two constants $C_2, D_2 > 0$ such that for any $x \in \mathcal{G}$ one has

$$\begin{aligned} -C_2\nu^{\frac{\alpha}{2}d_w/d_\alpha} &\leq \liminf_{t \rightarrow \infty} \frac{\log E_x[\exp(-\nu\mu(X_{[0,t]}^a))]}{t^{d_f/d_\alpha}} \\ &\leq \limsup_{t \rightarrow \infty} \frac{\log E_x[\exp(-\nu\mu(X_{[0,t]}^a))]}{t^{d_f/d_\alpha}} \leq -D_2\nu^{\frac{\alpha}{2}d_w/d_\alpha}. \end{aligned}$$

This is the gasket counterpart of the stable sausage asymptotics from [9] and of the ‘Wiener sausage’ asymptotics on nested fractals from [17].

2 Preliminaries

Notation. Throughout the paper A' will denote the complement of a set, and A^ρ — the open ρ -neighbourhood of a set. Generic numerical constants whose actual values are irrelevant for our purposes will be denoted by the lower case letter c . For important constants we will use lower case or capital letters with subscripts. An ‘admissible number’ is any number of the form 2^n , $n \in \mathbb{Z}$. When $A \subset \mathcal{G}$ is a measurable (Borel) set and (X_t) is a stochastic process, then

$$T_A = \inf\{t \geq 0 : X_t \in A\} \quad \text{and} \quad \tau_A = \inf\{t \geq 0 : X_t \notin A\}$$

denote respectively the entrance and the exit time of A .

2.1 The infinite Sierpiński gasket

The infinite Sierpiński gasket we will be working on is defined as a blowup of the unit gasket, which in turn is the unique fixed point of the hyperbolic iterated function system in \mathbb{R}^2 , consisting of three maps:

$$\phi_1(x) = \frac{x}{2}, \quad \phi_2(x) = \frac{x}{2} + \left(\frac{1}{2}, 0\right), \quad \phi_3(x) = \frac{x}{2} + \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right).$$

The unit gasket, $\mathcal{G}^{(0)}$, is the unique compact subset of \mathbb{R}^2 such that

$$\mathcal{G}^{(0)} = \phi_1(\mathcal{G}^{(0)}) \cup \phi_2(\mathcal{G}^{(0)}) \cup \phi_3(\mathcal{G}^{(0)}).$$

Let $V_{(0)} = \{a_1, a_2, a_3\} = \{(0, 0), (1, 0), (\frac{1}{2}, \frac{\sqrt{3}}{2})\}$ be the set of its vertices. Then we set:

$$\mathcal{G}^{(n)} = 2^n \mathcal{G}^{(0)} = ((\phi_1^{-1})^n(\mathcal{G}^{(0)}),$$

and

$$\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{G}^{(n)}.$$

Then inductively:

$$V_{(n+1)} = V_{(n)} \cup \{2^n a_1 + V_{(n)}\} \cup \{2^n a_2 + V_{(n)}\},$$

$$V^{(0)} = \bigcup_{n=0}^{\infty} V_{(n)}.$$

Elements of $V^{(0)}$ are exactly the vertices of all triangles of size 1 that build up the infinite gasket.

The gasket is equipped with the usual Euclidean metric inherited from the plane. Observe that in this metric one has $\mathcal{G}^{(M)} = B(0, 2^M)$. The set \mathcal{G} enjoys the scaling property

$$2\mathcal{G} = \mathcal{G},$$

By μ we denote the Hausdorff measure on \mathcal{G} in dimension $d_f = \frac{\log 3}{\log 2}$, normalized to have $\mu(\mathcal{G}^{(0)}) = 1$. The number d_f is called the fractal dimension of \mathcal{G} . The measure μ is a d_f -measure, i.e. there exist two positive constants a_1, a_2 such that for $r > 0, x \in \mathcal{G}$

$$a_1 r^{d_f} \leq \mu(B(x, r)) \leq a_2 r^{d_f}, \quad (2.1)$$

Another characteristic number of \mathcal{G} is its walk dimension, denoted $d_w(\mathcal{G})$ or just d_w . We have $d_w = \frac{\log 5}{\log 2}$. The spectral dimension of \mathcal{G} is by definition $d_s = \frac{2d_f}{d_w}$.

2.2 The Brownian motion

On the set \mathcal{G} one defines the Brownian motion (see [1, 2]), denoted by $(Z_t, P_x)_{t \geq 0, x \in \mathcal{G}}$. It is a symmetric, strong Markov, Feller process with continuous trajectories, whose distribution is invariant under local isometries of \mathcal{G} . It has a transition density with respect to the Hausdorff measure μ , denoted by $g(t, x, y)$. It is continuous in all its variables, symmetric in x, y . The following scaling property holds true:

$$g(t, 2x, 2y) = \frac{1}{2^{d_f}} g\left(\frac{t}{2^{d_w}}, x, y\right) = \frac{1}{3} g\left(\frac{t}{5}, x, y\right), \quad t > 0, \quad x, y \in \mathcal{G}, \quad (2.2)$$

where d_w is the walk dimension of \mathcal{G} . This transition density satisfies the following subgaussian estimates: there exist constants $a_3, a_4, a_5, a_6 > 0$ such that for $t > 0, x, y \in \mathcal{G}$ one has

$$\frac{a_3}{t^{d_s/2}} e^{-a_4 \left(\frac{|x-y|}{t^{1/d_w}} \right)^{\frac{d_w}{d_w-1}}} \leq g(t, x, y) \leq \frac{a_5}{t^{d_s/2}} e^{-a_6 \left(\frac{|x-y|}{t^{1/d_w}} \right)^{\frac{d_w}{d_w-1}}}.$$

In fact, the process in [2] is defined on a two-sided gasket, but it can be ‘folded’ to yield the process on the one-sided gasket we are working with.

2.3 Stable processes on the gasket, definition and relevant properties

Following [4, 7, 20], α -stable processes on \mathcal{G} are defined via subordination. Fix $\alpha \in (0, 2)$. Let S_t be the $\alpha/2$ -stable subordinator, independent of Z : the Lévy process on $[0, \infty)$ with Laplace transform $\mathbb{E}(e^{-uS_t}) = e^{-tu^{\alpha/2}}$; let $\eta_t(u)$, $t > 0, u \geq 0$ be the density of the distribution of S_t . Then we set

$$X_t = Z_{S_t}, \quad t \geq 0.$$

This process is called the *symmetric α -stable process* on \mathcal{G} . As $P[S_t = 0] = 0$, X has symmetric transition density given by

$$p(t, x, y) := \int_0^\infty \eta_t(u) g(u, x, y) du. \quad (2.3)$$

The transition density defined by (2.3) satisfies [4, Proposition 3.2]:

- (i1) $p(t, x, y)$ is jointly continuous in $(0, \infty) \times \mathcal{G} \times \mathcal{G}$,
- (i2) the semigroup of operators T_t with kernels $p(t, \cdot, \cdot)$ is both Feller and strong Feller,
- (i3) T_t is strongly continuous on $C_0(\mathcal{G})$.

Moreover, $p(t, x, y)$ fulfills the following estimate (see [4, Theorem 3.1] or [7, Theorem 1.1]): there exists a positive constant $A_0 = A_0(\mathcal{G}, \alpha)$ such that for $t > 0, x, y \in \mathcal{G}, x \neq y$,

$$\frac{1}{A_0} \min \left(\frac{t}{|x-y|^{d_\alpha}}, t^{-d_s/\alpha} \right) \leq p(t, x, y) \leq A_0 \min \left(\frac{t}{|x-y|^{d_\alpha}}, t^{-d_s/\alpha} \right), \quad (2.4)$$

where $d_s = 2d_f/d_w$ and $d_\alpha = d_f + \alpha d_w/2$, and also

$$\frac{1}{A_0} t^{-d_s/\alpha} \leq p(t, x, x) \leq A_0 t^{-d_s/\alpha}. \quad (2.5)$$

From the scaling of the transition density of the Brownian motion (2.2) and the scaling of the density of the subordinator (see e.g. [4, Formula 8])

$$\eta_t(u) = t^{-2/\alpha} \eta_1(t^{-2/\alpha} u), \quad t, u > 0, \quad (2.6)$$

we derive the following scaling property for the α -stable density:

$$p(t, 2x, 2y) = \frac{1}{2^{d_f}} p\left(\frac{t}{2^{(\alpha d_w/2)}}, x, y\right), \quad t > 0, x, y \in \mathcal{G}. \quad (2.7)$$

Indeed, one can write:

$$\begin{aligned}
p(t, 2x, 2y) &= \int_0^\infty g(u, 2x, 2y) \eta_t(u) \, du \\
&= \frac{1}{2^{d_f}} \int_0^\infty g\left(\frac{u}{2^{d_w}}, x, y\right) \eta_t(u) \, du \\
&= 2^{d_w - d_f} \int_0^\infty g(\tilde{u}, x, y) \eta_t(2^{d_w} \tilde{u}) \, d\tilde{u} \\
&= 2^{d_w - d_f} \int_0^\infty 2^{-d_w} g(u, x, y) \eta_{t/2^{(\alpha d_w/2)}}(u) \, du \\
&= \frac{1}{2^{d_f}} p\left(\frac{t}{2^{(\alpha d_w/2)}}, x, y\right) = \frac{1}{3} p\left(\frac{t}{5^{\alpha/2}}, x, y\right).
\end{aligned}$$

We will need the following estimate on exit time from balls:

Fact 2.1 [4, Lemma 4.3] *For each $k > 1$ there exists $A_1 = A_1(k)$ such that for $x \in \mathcal{G}$, $r > 0$, $y \in B(x, r/k)$ we have*

$$P_y[\tau_{B(x,r)} < t] \leq A_1 t r^{-\alpha d_w/2}. \quad (2.8)$$

Inequality (2.8) for $x = y$ gives the estimate for the supremum of the process:

$$\text{for any } x \in \mathcal{G}, \quad P_x\left[\sup_{0 \leq s \leq t} |X_s - X_0| > r\right] \leq A_1 t r^{-\alpha d_w/2}. \quad (2.9)$$

Let $U \subseteq \mathcal{G}$ be a bounded open set. By T_t^U we denote the L^2 -semigroup generated by the process killed on exiting U : for functions $f \in L^2(\mathcal{G}, \mu)$ one has $T_t^U f(x) = E_x[f(X_t); t < \tau_U]$.

Fact 2.2 [4, Proposition 3.2] *The semigroup $(T_t^U)_{t \geq 0}$ has both Feller and strong Feller properties.*

The scaling of the transition density results in the following scaling for the principal eigenvalue $\lambda(U)$ (i.e. the smallest eigenvalue of the generator of the process killed outside U):

$$\lambda(2U) = \frac{1}{2^{\alpha d_w/2}} \lambda(U) = \frac{1}{5^{\alpha/2}} \lambda(U). \quad (2.10)$$

The harmonic measure of an open set is defined classically.

Definition 2.1 *Let $U \subset \mathcal{G}$ be open and nonempty, let $x \in U$. The P_x -distribution of X_{τ_U} is called the harmonic measure of U .*

If U is nonempty and bounded then the distribution of X_{τ_U} is absolutely continuous with respect to μ on $\text{int}(U')$ (see [4, p. 178]). Its density is called the Poisson kernel and denoted by $P_U(x, y)$. We have the following estimates for the Poisson kernels of balls.

Fact 2.3 [4, Proposition 6.4] *Let There exists a constant $A_2^0 > 0$ such that for each $k > 1$, $x_0 \in \mathcal{G}$, $r > 0$ and for $A_2 = c(\frac{k+1}{k-1})^{d_\alpha} A_2^0$, $\tilde{A}_2 = c(\frac{k-1}{k+1})^{d_\alpha} A_2^0$, we have*

$$P_{B(x_0, r)}(x, z) \leq A_2 r^{\alpha d_w/2} |x - z|^{-d_\alpha}, \quad x \in B(x_0, r), \quad z \in B(x_0, kr)', \quad (2.11)$$

$$P_{B(x_0, r)}(x, z) \geq \tilde{A}_2 r^{\alpha d_w/2} |x - z|^{-d_\alpha}, \quad x \in B(x_0, r/k), \quad z \in \text{Int}(B(x_0, r)'). \quad (2.12)$$

3 The integrated density of states for stable processes on the gasket evolving among killing Poissonian obstacles

Let $\nu > 0$ (the intensity) and $a > 0$ (the radius of the obstacles) be fixed. Consider the Poisson point process \mathcal{N} with intensity $\nu\mu$ on \mathcal{G} , denote by $(\Omega, \mathcal{M}, \mathbb{Q})$ the probability space it is defined on. A ball with radius a ('an obstacle') is attached at each of the Poisson points. One denotes: $\mathcal{N}(\omega) = \{x_i\}_{i \in \mathbb{N}}$, $\mathcal{N}_a(\omega) = \bigcup_i \overline{B}(x_i, a)$, $\mathcal{N}(A) = \#\{x_i \in \mathcal{N}(\omega) : x_i \in A\}$. The set $\mathcal{N}_a(\omega)$ is called the obstacle set, and the set $\mathcal{O}(\omega) = \mathcal{G} \setminus \mathcal{N}_a(\omega)$ – the free open set. We assume that the stable process and the Poisson process are independent. The stable process evolves in $\mathcal{O}(\omega)$ and is killed when it jumps to the obstacle set $\mathcal{N}_a(\omega)$.

To define the integrated density of states for such a system, one considers the stable process on a bounded gasket $\mathcal{G}^{(M)}$, $M = 1, 2, \dots$, killed when it enters the obstacle set, or when it jumps out of the interior of $\mathcal{G}^{(M)}$. Formally speaking, such a process should be denoted by $X^{(M, \mathcal{N}_a)}$, but for the sake of notation we will denote it just by X . It can be realized in the space of càdlàg functions and its transition density (with respect to the Hausdorff measure on \mathcal{G}) can be expressed by the usual Dynkin-Hunt formula

$$p^{M, \omega}(t, x, y) = p(t, x, y) - \mathbb{E}_x[p(t - T_{M, \omega}, X_{T_{M, \omega}}, y) \mathbf{1}_{T_{M, \omega} < t}],$$

where

$$T_{M, \omega} = \inf\{t \geq 0 : X_t \in \bigcup_i \overline{B}(x_i, a) \text{ or } X_t \notin \text{Int } \mathcal{G}^{(M)}\}$$

denotes the entrance time into the obstacle set or into the closure of $(\mathcal{G}^{(M)})'$. The transition density $p^{M, \omega}(t, x, y)$ has a more convenient representation:

$$p^{M, \omega}(t, x, y) = \begin{cases} p(t, x, y) P_{x, y}^t[T_{M, \omega} > t] & \text{for } x, y \in \text{Int } \mathcal{G}^{(M)} \cap \mathcal{O}(\omega), \\ 0 & \text{else.} \end{cases} \quad (3.1)$$

$P_{x, y}^t$ are the bridge measures: conditional distributions of the process subject to the condition $X_0 = x, X_t = y$. The continuity of $p(\cdot, \cdot, \cdot)$ in time and space variables makes these bridges well-defined [6, Theorem 1]; we also refer to that paper for more information on Markovian bridges. Similarly as in [15, Proposition 4.2], we see that the expression (3.1) defines \mathbb{Q} -a.s. a transition density which is symmetric in x, y .

In virtue of the representation (3.1) and the estimate (2.5), we see that the semigroup on $L^2(\mathcal{G}^{(M)}, \mu)$, associated with kernels $p^{M, \omega}$, denoted $(T_t^{M, \omega})_{t \geq 0}$, consists of self-adjoint trace-class operators, so its generator $\mathcal{L}^{M, \omega}$ is self-adjoint and has pure point spectrum consisting of nonnegative eigenvalues without accumulation points:

$$0 \leq \lambda_1(M, \omega) \leq \lambda_2(M, \omega) \leq \dots \leq \lambda_n(M, \omega) \leq \dots \quad (3.2)$$

One considers then the empirical measures with atoms at points of these spectra, normalized by the volume of the sets $\mathcal{G}^{(M)}$:

$$l(M, \omega) = \frac{1}{\mu(\mathcal{G}^{(M)})} \sum_{n=1}^{\infty} \delta_{\{\lambda_n(M, \omega)\}}, \quad (3.3)$$

and we are interested in the asymptotical behavior of those measures as $M \rightarrow \infty$.

As in the classical case, these measures have a vague, nonrandom limit l . This limiting measure is called *the integrated density of states* for the α -stable process, or the α -stable integrated density of states (α -IDS, for short). More precisely, in the paper [11] we have proven the following.

Theorem 3.1 [11, Theorem 3.3] *Almost surely with respect to the measure \mathbb{Q} , the measures $l(M, \omega)$ converge vaguely to a nonrandom measure l on $[0, \infty)$.*

The key to the method is the following representation of the Laplace transform of empirical measures (3.3):

$$\begin{aligned}
L(M, \omega)(t) &= \int_0^\infty e^{-\lambda t} dl(M, \omega)(t) \\
&= \frac{1}{\mu(\mathcal{G}^{(M)})} \sum_{n=1}^\infty e^{-\lambda_n(M, \omega)t} = \frac{1}{\mu(\mathcal{G}^{(M)})} \text{Tr } T_t^{M, \omega} \\
&= \frac{1}{\mu(\mathcal{G}^{(M)})} \int_{\mathcal{G}^{(M)}} p^{M, \omega}(t, x, x) d\mu(x) \\
&= \frac{1}{\mu(\mathcal{G}^{(M)})} \int_{\mathcal{G}^{(M)}} p(t, x, x) P_{x, x}^t[T_{M, \omega} > t] d\mu(x). \tag{3.4}
\end{aligned}$$

The statement of Theorem 3.1 was achieved by proving that for any $t > 0$ the averaged Laplace transforms converge to $L(t)$:

$$\mathbb{E}_{\mathbb{Q}}[L(M, \omega)(t)] =: L_M(t) \xrightarrow{M \rightarrow \infty} L(t) \tag{3.5}$$

and that the measure l with Laplace transform L is the \mathbb{Q} -almost sure vague limit of the measures $l(M, \omega)$. Let us note that for the Brownian motion on fractals the limit in (3.5) was monotone increasing, what followed from symmetries of the process (see [17]), therefore to get the convergence of $L_M(t)$ when $M \rightarrow \infty$ one just had to find an upper bound. For stable processes, such symmetries are no longer true, we cannot use monotonicity, and the proof of the convergence got substantially more difficult.

4 Asymptotics for the IDS and the stable sausage

As indicated in the Introduction, the behaviour of the IDS when $\lambda \rightarrow 0^+$ and the asymptotics of its Laplace transform at $+\infty$ are linked via a Tauberian-type theorem, therefore it is enough to get bounds on $L(t)$ when $t \rightarrow \infty$. In the fractal setting, such bounds were previously obtained for the Brownian motion on the Sierpiński gasket with Poissonian obstacles [16]. We also refer to [18] for a direct proof of the Lifschitz tail for the Brownian motion on nested fractals with potential interaction. Let us note that the methods we use are also suitable for determining the ‘sausage asymptotics’ when $t \rightarrow \infty$. This topic for the Brownian motion on the Sierpiński gasket was previously addressed in [16], and for general nested fractals – in [17]. Similarly to the Brownian motion case, the α -IDS asymptotics and the α -stable sausage asymptotics are the same up to a constant, although neither seems to be a direct consequence of the other.

The lower and the upper bounds for the Laplace transform are obtained separately. The lower bound estimate is easier: adjusting the ideas from [17] to the case where one cannot use the monotonicity of expressions approximating $L(t)$, we get the desired result by imposing some additional conditions on the process and on the cloud. The proof of the matching upper bound uses a non-diffusive counterpart of Sznitman’s theorem [21, Theorem 1.3], obtained in [12, Theorem 1] (see also Theorem 4.2 below).

4.1 The lower bound

As usual [16, 21], to get the lower bound observe that the event $\{T_{M,\omega} > t\}$ holds true if the process stays in a sufficiently large ball up to time t , and no Poisson points are present in the vicinity of this ball. For the sausage estimate, one just picks a large ball centered at the origin. For the IDS estimate, the ball will depend on the starting point. The estimates are then obtained via semigroup methods.

Theorem 4.1 *There exist constants $C_1, C_2 > 0$ such that for the Laplace transform of the α -IDS one has:*

$$\liminf_{t \rightarrow \infty} \frac{\log L(t)}{t^{d_f/d_\alpha}} \geq -C_1 \nu^{\frac{\alpha}{2} d_w/d_\alpha} \quad (4.1)$$

and for the α -stable sausage volume one has, for any $x \in \mathcal{G}$,

$$\liminf_{t \rightarrow \infty} \frac{\log E_x[\exp(-\nu \mu(X_{[0,t]}^a))]}{t^{d_f/d_\alpha}} \geq -C_2 \nu^{\frac{\alpha}{2} d_w/d_\alpha}. \quad (4.2)$$

Proof of (4.1). Let $t > 0$ be fixed. As $L(t) = \lim_{M \rightarrow \infty} L_M(t)$, it is enough to find a lower bound on $L_M(t)$, independent of $M > M_0(t)$. Recall that $d_\alpha = d_f + \alpha d_w/2$ and let

$$M_0 = M_0(t) = \left\lceil \frac{1}{d_\alpha} \frac{\log(t/\nu)}{\log 2} \right\rceil \quad (4.3)$$

($[x]$ denotes the biggest integer not exceeding x). This is the unique integer for which

$$2^{M_0} \leq \left(\frac{t}{\nu}\right)^{1/d_\alpha} < 2^{M_0+1}. \quad (4.4)$$

Assuming $M > M_0$, write

$$\begin{aligned} L_M(t) &= \frac{1}{\mu(\mathcal{G}^{\langle M \rangle})} \int_{\mathcal{G}^{\langle M \rangle}} p(t, x, x) P_{x,x}^t \otimes \mathbb{Q}[T_{M,\omega} > t] d\mu(x) \\ &= \frac{1}{\mu(\mathcal{G}^{\langle M \rangle})} \sum_{\mathcal{T}} \int_{\mathcal{T}} p(t, x, x) P_{x,x}^t \otimes \mathbb{Q}[T_{M,\omega} > t] d\mu(x), \end{aligned}$$

where the sum is taken over all the triangles of size 2^{M_0} building $\mathcal{G}^{\langle M \rangle}$. These triangles have disjoint interiors and there are 3^{M-M_0} of them. Choose \mathcal{T} to be one of those triangles. When $x \in \mathcal{T}$, then the event $\{T_{M,\omega} > t\}$ will hold when the process stays in \mathcal{T} up to time t and there are no obstacles in \mathcal{T}^a . Consequently, we have:

$$\begin{aligned} L_M(t) &\geq \frac{1}{\mu(\mathcal{G}^{\langle M \rangle})} \sum_{\mathcal{T}} \int_{\mathcal{T}} p(t, x, x) P_{x,x}^t \otimes \mathbb{Q}[\tau_{\mathcal{T}} > t, \mathcal{N}(\mathcal{T}^a) = 0] d\mu(x) \\ &= \frac{1}{\mu(\mathcal{G}^{\langle M \rangle})} \sum_{\mathcal{T}} \left[\int_{\mathcal{T}} p(t, x, x) P_{x,x}^t [\tau_{\mathcal{T}} > t] d\mu(x) \right] \cdot \mathbb{Q}[\mathcal{N}(\mathcal{T}^a) = 0] \\ &= \frac{1}{\mu(\mathcal{G}^{\langle M \rangle})} \sum_{\mathcal{T}} \text{Tr } T_t^{\mathcal{T}} \cdot e^{-\nu \mu(\mathcal{T}^a)}, \end{aligned} \quad (4.5)$$

where $(T_t^{\mathcal{T}})_{t \geq 0}$ is the Dirichlet stable semigroup on \mathcal{T} . Clearly, $\text{Tr } T_t^{\mathcal{T}} \geq e^{-t\lambda(\mathcal{T})}$, $\lambda(\mathcal{T})$ being the principal eigenvalue of \mathcal{T} (relative to the stable process). It is a classical fact (see e.g. [8, Theorem 3.4]) that

$$\lambda(\mathcal{T}) \leq (\lambda^{BM}(\mathcal{T}))^{\alpha/2}$$

where $\lambda^{BM}(\mathcal{T})$ is the principal Brownian Dirichlet eigenvalue of \mathcal{T} . From symmetry properties of the Brownian motion on the gasket we see that for any triangles $\mathcal{T}, \mathcal{T}'$ appearing in the sum above one has $\lambda^{BM}(\mathcal{T}) = \lambda^{BM}(\mathcal{T}') = \lambda^{BM}(\mathcal{G}^{\langle M_0 \rangle})$. We also have $\mu(\mathcal{T}^a) \leq \mu(\mathcal{T}) + ca^{d_f}$, with c – a numerical constant. Inserting these bounds into (4.5) we get

$$\begin{aligned} L_M(t) &\geq \frac{1}{\mu(\mathcal{G}^{\langle M \rangle})} 3^{M-M_0} e^{-t(\lambda^{BM}(\mathcal{G}^{\langle M_0 \rangle}))^{\alpha/2}} \cdot e^{-\nu(3^{M_0} + ca^{d_f})} \\ &= \frac{1}{3^{M_0}} e^{-t(\lambda^{BM}(\mathcal{G}^{\langle M_0 \rangle}))^{\alpha/2} - \nu(3^{M_0} + ca^{d_f})}. \end{aligned}$$

The scaling of the Brownian principal eigenvalue ($\lambda^{BM}(2U) = \frac{1}{5}\lambda^{BM}(U)$) gives $\lambda^{BM}(\mathcal{G}^{\langle M_0 \rangle}) = \frac{1}{5^{M_0}}\lambda^{BM}(\mathcal{G}^{\langle 0 \rangle})$. Moreover, as $3^{M_0} = 2^{d_f M_0}$ and $5^{M_0} = 2^{d_w M_0}$, from (4.4) we obtain

$$5^{M_0} > \frac{1}{5} \left(\frac{t}{\nu} \right)^{d_w/d_\alpha} \quad \text{and} \quad 3^{M_0} \leq \left(\frac{t}{\nu} \right)^{d_f/d_\alpha},$$

so that

$$\begin{aligned} t(\lambda^{BM}(\mathcal{G}^{\langle M_0 \rangle}))^{\alpha/2} + \nu 3^{M_0} &\leq t^{d_f/d_\alpha} \nu^{\frac{\alpha}{2} d_w/d_\alpha} \left(5^{\alpha/2} (\lambda^{BM}(\mathcal{G}^{\langle 0 \rangle}))^{\alpha/2} + 1 \right) \\ &=: C_1 t^{d_f/d_\alpha} \nu^{\frac{\alpha}{2} d_w/d_\alpha}, \end{aligned}$$

i.e.

$$L_M(t) \geq e^{-C_1 t^{d_f/d_\alpha} \nu^{\frac{\alpha}{2} d_w/d_\alpha}} \cdot e^{-\nu ca^{d_f}} \cdot \left(\frac{\nu}{t} \right)^{d_f/d_\alpha}. \quad (4.6)$$

This bound is independent of t and valid as long as $M > M_0(t)$, therefore it is also true for $L(t) = \lim_{M \rightarrow \infty} L_M(t)$. Taking the logarithm, dividing by t^{d_f/d_α} and passing to the limit we obtain (4.1).

Proof of (4.2). Let $t > 0$ be given and large enough to have $x \in \text{Int } \mathcal{G}^{\langle M_0 \rangle}$, where $M_0 = M_0(t)$ is introduced in (4.3). Recalling that we have denoted $\mathcal{O}(\omega) = \mathcal{G} \setminus \mathcal{N}_a(\omega)$, we can write

$$E_x \left[\exp[-\nu \mu(X_{[0,t]}^a)] \right] = P_x \otimes \mathbb{Q}[\tau_{\mathcal{O}(\omega)} > t].$$

As in the proof of (4.1) we observe that the event $[\tau_{\mathcal{O}(\omega)} > t]$ will hold if the process stays in the ball $B(0, 2^{M_0}) = \mathcal{G}^{\langle M_0 \rangle}$ up to time t and the a -vicinity of this ball receives no Poisson points. It follows

$$E_x \left[\exp[-\nu \mu(X_{[0,t]}^a)] \right] \geq \exp[-\nu \mu((\mathcal{G}^{\langle M_0 \rangle})^a)] P_x[\tau_{\mathcal{G}^{\langle M_0 \rangle}} > t].$$

Now write $x = 2^{M_0} y$, then scale down using (2.7) and get

$$P_x[\tau_{\mathcal{G}^{\langle M_0 \rangle}} > t] = P_y[\tau_{\mathcal{G}^{\langle 0 \rangle}} > \frac{t}{5^{M_0 \alpha/2}}] \geq P_y[\tau_{\mathcal{G}^{\langle 0 \rangle}} > 5^{\alpha/2} t^{d_f/d_\alpha} \nu^{\frac{\alpha}{2} d_w/d_\alpha}].$$

The rest of the proof goes identically as that of [17, Theorem 2.1]. \square

4.2 The upper bound

We intend to use the Sznitman's ‘enlargement of obstacles’ method in its non-diffusive version from [12]. The method works for processes with compact state-space, so the first ingredient needed in the proof is the reduction of the problem to a one with a compact state-space. Once it is done, the method relies on replacing the microscopic Poisson obstacles with bigger balls of ‘intermediate’ size and on controlling the possible increase of principal eigenvalue when the process is killed on entering those bigger obstacles.

To make the paper self-contained, we briefly describe the method (Section 4.2.1), then we carry out the reduction to the compact problem (Section 4.2.2) and prove the necessary estimates for the ‘stable process on the compact set’ (Section 4.2.3). Finally we enlarge the obstacles – the conclusion of the proof is much alike that in the Brownian motion case (Section 4.2.4).

4.2.1 Description of the method

The method we are going to use works in the following situation:

- (Ξ, d, μ) is a compact metric measure space equipped with a doubling probability measure μ charging all open balls. More precisely, we assume that there exist constants $\kappa > 0$ and $R_0 > 0$ such that for any ball $B(x, r)$, $0 < r < R_0$ one has

$$\mu(\overline{B}(x, r/3)) \geq \kappa^{-1} \mu(\overline{B}(x, r)). \quad (4.7)$$

- (ξ_t) is a symmetric strong Markov, Feller process on Ξ with càdlàg trajectories and transition density $p(\cdot, \cdot, \cdot)$, regular enough to have well-defined symmetric bridge measures $P_{x,y}^t$.

Suppose that points $x_i \in \Xi$, $i = 1, \dots, n$ are given, together with positive constants $a > 0, \epsilon > 0$. Closed balls $\overline{B}(x_i, a\epsilon)$ $i = 1, \dots, n$, are considered fixed and we call them ‘obstacles’ – the process is killed when it enters one of those balls.

Points x_i are labeled ‘good’ or ‘bad’ according to the following rule.

Let $R > 0$, $b \gg a$ and $\delta > 0$ be given. A point x_i is called (R, b, δ) –good, or just *good*, if for every set $F = B(x_i, 10\epsilon b R^l)$ such that $10\epsilon b R^l \leq R_0$, $l \in \mathbb{Z}_+$, we have

$$\mu\left(\bigcup_{j=1}^N B(x_j, b\epsilon) \cap F\right) \geq \frac{\delta}{\kappa} \mu(F).$$

Otherwise, x_i is called (R, b, δ) –bad (or just *bad*).

Denote $\Theta := \Xi \setminus \bigcup_{i=1}^n \overline{B}(x_i, a\epsilon)$ and for $b > a$, $\Theta_b := \Xi \setminus \bigcup_{i: x_i \text{--good}} \overline{B}(x_i, b\epsilon)$, then by λ_Θ (resp. λ_b) – the principal eigenvalue of the generator of the processes killed on exiting Θ (resp. on exiting Θ_b).

Below we list the properties of the process which need to be established in order to make the method work. The numbers $a, b, \epsilon, \delta > 0$ are fixed (same as above).

We assume that there exist an exponent $s > 0$ and numbers $R > 3$, $c_1, c_2, c_3, c_4 > 0$ such that:

- (A1)** for all $x, y \in \Xi$ with $d(x, y) \leq \beta$, where β is an arbitrary number such that $10b\epsilon \leq \beta < \frac{R_0}{R}$, and for every compact set E , which satisfies

$$\frac{\mu(E \cap \overline{B}(y, \beta))}{\mu(\overline{B}(y, \beta))} \geq \frac{\delta}{\kappa},$$

we have:

$$P_x[T_E < \tau_{B(y, R\beta)}] \geq c_1; \quad (4.8)$$

- (A2)** with β – as above, whenever $x \in \Xi$ satisfies $d(x_i, x) \leq \epsilon b < R_0$ for some $i \in \{1, \dots, n\}$, then one has:

$$P_x(T_{\overline{B}(x_i, a\epsilon)} \leq \frac{\epsilon^s}{2}) \geq 2c_2; \quad (4.9)$$

- (A3)** for all $x, y \in \Xi$, when $d(x, y) = b\epsilon$ then

$$P_y(\tau_{B(x, 10R\epsilon b)} < \frac{\epsilon^s}{2}) < c_2, \quad (4.10)$$

where c_2 is the constant from **(A2)**;

(A4) there exists a decreasing function $\phi : (0, \infty) \rightarrow (0, 1]$ such that if $d(x, y) \leq \epsilon r < R_0$ then

$$P_x(T_{B(y, \epsilon b)} \leq \frac{\epsilon^s}{2}) \geq \phi(r); \quad (4.11)$$

(A5) there exists a constant $c_3 > 0$ such that for all $0 < r < R_0$, $y \in \Xi$ and $x \in B(y, r)$, when $\rho > 3r$, then we have

$$P_x(\xi_{\tau_{B(y, r)}} \in B(y, \rho)') \leq c_3 \left(\frac{r}{\rho}\right)^s; \quad (4.12)$$

(A6) for all $x, y \in \Xi$ one has

$$p(1, x, y) \leq c_4. \quad (4.13)$$

The main theorem of [12] is the following.

Theorem 4.2 *Let a, b, ϵ be as above and let $K, \delta > 0$ be given. Suppose that (A1)–(A6) are satisfied with a number $R > 3$ that satisfies*

$$\frac{c_3}{R^s - 1} \leq \frac{1}{2}(e^K(1 + c_4(1 + K/\delta)))^{-1} \quad (4.14)$$

(the exponent s and the constants c_3, c_4 come from the assumptions above). Then there exists

$$\epsilon_0 = \epsilon_0(b, K, \delta, R_0, R, c_1, c_2, c_3, c_4, s, \phi(\cdot)) > 0$$

such that for $\epsilon < \epsilon_0$ one has

$$\lambda_b \wedge K \leq \lambda_\Theta \wedge K + \delta. \quad (4.15)$$

In fact, if we can find a number $R > 3$ for which the assumptions (A1)–(A5) hold true, and R' is a number that satisfies (4.14), then the assumptions are fulfilled for $\bar{R} = R \vee R'$ as well, without change in other constants.

4.2.2 The projected process

We want to have a process on the finite gasket $\mathcal{G}^{(0)}$, locally behaving as the stable process, and with infinite lifetime. To this goal, we ‘project’ the unrestricted stable process on \mathcal{G} onto $\mathcal{G}^{(0)}$, using the projection $\pi_0 : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ from [16, Section 5.2].

Projected processes derived from subordinate Brownian motions were considered in [11] and they were used there for proving the existence of the density of states for subordinate processes. Stable processes on \mathcal{G} fall within this category. In particular, the projected α –stable process on $\mathcal{G}^{(0)}$ is a strong Markov and Feller process with continuous, symmetric transition density. See [11, Section 2.2.3].

We recall briefly its definition.

Following [16], we put labels on vertices from $V^{(0)}$. We have $V^{(0)} \subset \mathbb{Z}e_1 + \mathbb{Z}e_2$, where $e_1 = (1, 0)$, $e_2 = (1/2, \sqrt{3}/2)$. Consider the commutative 3–group \mathbb{A}_3 consisting of even permutations of 3 elements $\{u, v, w\}$, i.e. $\mathbb{A}_3 = \{id, p_1, p_2\}$, where $p_1 = (u, v, w)$, $p_2 = (u, w, v)$. With every point $x = ne_1 + me_2$ we associate the permutation $p_1^n \circ p_2^m \in \mathbb{A}_3$. In particular, a permutation is assigned to every point $x \in V^{(0)}$ and the label of x is its value at u , i.e. $p_1^n \circ p_2^m(u)$.

After the vertices have been labeled, we define the projection. Every nonlattice point $x \in \mathcal{G} \setminus V^{(0)}$ belongs to exactly one triangle of size 1, and can be written as

$$x = x_u u(x) + x_v v(x) + x_w w(x),$$

where $u(x), v(x), w(x)$ are the points of $\Delta_0(x)$ with respective labels u, v, w , and numbers $x_u, x_v, x_w \in (0, 1)$ satisfy $x_u + x_v + x_w = 1$. For such a point we set

$$\pi_0(x) := x_u \cdot u(0) + x_v \cdot v(0) + x_w \cdot w(0),$$

where we have denoted $u(0) = (0, 0)$, $v(0) = (1/2, \sqrt{3}/2)$, $w(0) = (0, 1)$. When $x \in V^{(0)}$, then x itself has a label and we map it to the vertex of $\mathcal{G}^{(0)}$ with corresponding label.

Then we define

$$Z_t^{(0)} := \pi_0(X_t) \quad (4.16)$$

and we call this process the *projected stable process on $\mathcal{G}^{(0)}$* .

In analogy to the reflected Brownian motion from [16] whose transition density is given by

$$g^{(0)}(t, x, y) = \begin{cases} \sum_{y' \in \pi_0^{-1}(y)} g(t, x, y') & \text{if } x, y \in \mathcal{G}^{(0)}, y \notin V^{(0)} \setminus \{(0, 0)\}, \\ 2 \sum_{y' \in \pi_0^{-1}(y)} g(t, x, y') & \text{if } y \in V^{(0)} \setminus \{(0, 0)\} \end{cases}$$

the projected stable process (4.16) has transition density $p^{(0)}(t, x, y)$ given by:

$$p^{(0)}(t, x, y) = \begin{cases} \sum_{y' \in \pi_0^{-1}(y)} p(t, x, y') & \text{if } x, y \in \mathcal{G}^{(0)}, y \notin V^{(0)} \setminus \{(0, 0)\} \\ 2 \sum_{y' \in \pi_0^{-1}(y)} p(t, x, y') & \text{if } y \in V^{(0)} \setminus \{(0, 0)\}. \end{cases} \quad (4.17)$$

Probabilities related to the projected process will be denoted by $(P_x^{(0)})_{x \in \mathcal{G}^{(0)}}$.

It is immediate to see that the projection commutes with subordination, i.e.

$$p^{(0)}(t, x, y) = \int_0^\infty g^{(0)}(u, x, y) \eta_t(u) du \quad (4.18)$$

and not hard to prove [11, Lemma 2.4] that for given $t > 0$ the series $\sum_{y' \in \pi_0^{-1}(y)} p(t, x, y')$ is uniformly convergent with respect to $x, y \in \mathcal{G}^{(0)}$. The function $p^{(0)}$ inherits symmetry and continuity properties of $g^{(0)}$, established in [16]. The following estimate can be deduced from [11, Lemma 2.5]: there exists a constant $A_3 > 0$ such that for $t > 0, x, y \in \mathcal{G}^{(0)}$ one has:

$$p^{(0)}(t, x, y) \leq A_3(t^{-2d_f/\alpha_{dw}} \vee 1). \quad (4.19)$$

The continuity properties of $p^{(0)}$ yield the Feller property and then the strong Markov property of the projected process. Also, these conditions are sufficient for defining bridge measures related to the projected process (see [6, Theorems 1,2]). The bridge measure relative to the projected process on $[0, t]$, starting from x and conditioned to arrive at point y at time t will be denoted by $Q_{x,y}^{t,(0)}$.

The following proposition permits to relate the bridge of the projected process to the bridge of the free process.

Proposition 4.1 (i) *Let $x, y \in \mathcal{G}$ be two points in the same 0-fiber, i.e. $\pi_0(x) = \pi_0(y)$. Then the measures $\pi_0(P_x)$ and $\pi_0(P_y)$ on $D([0, t], \mathcal{G}^{(0)})$ coincide. Moreover for every $z \in \mathcal{G}^{(0)}$ and x, y as above we have:*

$$\sum_{z' \in \pi_0^{-1}(z)} p(t, x, z') = \sum_{z' \in \pi_0^{-1}(z)} p(t, y, z').$$

(ii) *For $x, y \in \mathcal{G} \setminus V^{(0)}$, the image under π_0 on $D([0, t], \mathcal{G}^{(0)})$ of the measure*

$$\sum_{y' \in \pi_0^{-1}(\pi_0(y))} p(t, x, y') P_{x,y'}^t[\cdot]$$

is equal to $q^0(t, \pi_0(x), \pi_0(y))Q_{\pi_0(x), \pi_0(y)}^{t, \langle 0 \rangle}[\cdot]$.

In particular we have, for all $A \in \mathcal{B}(D[0, t], \mathcal{G}^{(0)})$

$$q^0(t, \pi_0(x), \pi_0(y))Q_{\pi_0(x), \pi_0(y)}^{t, \langle 0 \rangle}[A] = \sum_{y' \in \pi_0^{-1}(\pi_0(y))} p(t, x, y')P_{x, y'}^t[\pi_0^{-1}(A)] \quad (4.20)$$

Proof. These properties follow from their counterparts for the Brownian motion [16, Theorem 3 and Lemma 8] and the subordination formula (4.18). See also [11, Lemma 2.6]. \square

4.2.3 Recurrence properties for α -stable processes in fractals

As we would like to use Theorem 4.2 for $\Xi = \mathcal{G}^{(0)}$, $\xi = X^{(0)}$, we need to establish the relevant recurrence properties of the projected stable process on $\mathcal{G}^{(0)}$.

The x^{d_f} -Hausdorff measure on $\mathcal{G}^{(0)}$, as well as on \mathcal{G} , is a d_f -measure and as such is doubling. We also know that the projected α -stable process is strong Markov, Feller, symmetric and regular enough to have well-defined bridge measures.

Proposition 4.2 *Let $X_t^{(0)}$ be the reflected α -stable process on $\mathcal{G}^{(0)}$ defined by (4.16). Let the numbers $b > a > 0$, $\epsilon > 0$, and $\delta > 0$ be fixed. Then (A1)–(A6) are satisfied, with $s = (\alpha d_w)/2$.*

Proof. We first check assumptions (A1)–(A5) for the α -stable process on the infinite fractal. Since for every Borel set $A \subset \mathcal{G}^{(0)}$ and for every $x \in A$ one has $P_x^{(0)}(\tau_A > t) \geq P_x(\tau_A > t)$, and for $y \in \mathcal{G}^{(0)} \setminus A$ one has $P_y^{(0)}(T_A \leq t) \geq P_y(T_A \leq t)$, conditions (A1)–(A5) for the process on the unbounded fractal will yield respective properties for the processes on the unit fractal.

Proof of (A1). For given $y \in \mathcal{G}$, suppose E satisfies $\frac{\mu(E \cap \overline{B}(y, \beta))}{\mu(\overline{B}(y, \beta))} \geq \frac{\delta}{\kappa}$ and $|x - y| \leq \beta$. Is R is large enough (say, $R > 10$) we have

$$\begin{aligned} P_x[T_E < \tau_{B(y, R\beta)}] &\geq P_x[T_{E \cap B(y, \beta)} < \tau_{B(y, R\beta)}] \\ &\geq \inf_u P_x[X_{\tau_{B(y, 2\beta)}} \in B(u, \beta), X_{\tau_{B(y, 2\beta)} + \tau_{B(u, \beta)} \circ \theta_{\tau_{B(y, 2\beta)}}} \in E \cap B(y, \beta)], \end{aligned} \quad (4.21)$$

where the infimum is taken over $\{u \in \mathcal{G} : 4\beta < |y - u| < 6\beta\}$. From the strong Markov property applied at the stopping time $\tau_{B(y, 2\beta)}$ we can estimate (4.21) from below by

$$\inf_{\{u \in \mathcal{G} : 4\beta < |y - u| < 6\beta\}} \left(P_x[X_{\tau_{B(y, 2\beta)}} \in B(u, \beta)] \inf_{z \in B(u, \beta)} P_z[X_{\tau_{B(u, \beta)}} \in E \cap B(y, \beta)] \right).$$

For all $u \in B(y, 6\beta) \setminus B(y, 4\beta)$ one has $B(u, \beta) \subset B(y, 3\beta)'$. Using this fact and the explicit estimate on the Poisson kernel (2.12) (with $x_0 = y$, $r = 2\beta$, and $k = 2$) we get:

$$\begin{aligned} P_x[X_{\tau_{B(y, 2\beta)}} \in B(u, \beta)] &\geq \tilde{A}_2 \int_{B(u, \beta)} \frac{(2\beta)^{\alpha d_w/2}}{|x - z|^{d_\alpha}} d\mu(z) \\ &\geq c\beta^{\alpha d_w/2} \beta^{-d_\alpha} \mu(B(u, \beta)) \geq ca_1 =: c_0. \end{aligned}$$

Similarly, using additionally the assumption on the measure of $E \cap \overline{B}(y, \beta)$ (which is the same as the measure of $E \cap B(y, \beta)$),

$$\begin{aligned} \inf_{z \in B(u, \beta)} P_z[X_{\tau_{B(u, \beta)}} \in E \cap B(y, \beta)] &\geq \tilde{A}_2 \inf_{z \in B(u, \beta)} \int_{E \cap B(y, \beta)} \frac{(2\beta)^{\alpha d_w/2}}{|\zeta - z|^{d_\alpha}} d\mu(\zeta) \\ &\geq c\beta^{\alpha d_w/2} \beta^{-d_\alpha} \mu(E \cap B(y, \beta)) \\ &= c\beta^{\alpha d_w/2} \beta^{-d_\alpha} \frac{\delta}{\kappa} \beta^{d_f} = \frac{c\delta}{\kappa} =: c'_0. \end{aligned}$$

Observe that the constants c_0 and c'_0 do not depend on β . Therefore

$$P_x[T_E < \tau_{B(y, R\beta)}] \geq c_0 c'_0 =: c_1.$$

This completes the proof of **(A1)**.

Proof of (A2). When $|x - x_i| \leq \epsilon b < R_0$, then (triangle inequality) for $y \in B(x_i, a\epsilon)$ one has $|y - x| \leq \epsilon(b + a) \leq 2\epsilon b$, and one can proceed as follows:

$$\begin{aligned} P_x(T_{B(x_i, a\epsilon)} \leq \frac{1}{2} \epsilon^{\alpha d_w/2}) &\geq P_x(X_{\frac{1}{2} \epsilon^{\alpha d_w/2}} \in B(x_i, a\epsilon)) \\ &= \int_{B(x_i, a\epsilon)} p(\epsilon^{\alpha d_w/2}/2, x, y) d\mu(y) \\ &\geq \mu(B(x_i, a\epsilon)) \inf_{y \in B(x_i, a\epsilon)} \frac{1}{A_0} \min\left(\frac{\epsilon^{\alpha d_w/2}}{2|x-y|^{d_\alpha}}, \left(\frac{\epsilon^{\alpha d_w/2}}{2}\right)^{-d_s/\alpha}\right) \\ &\geq a_1(a\epsilon)^{d_f} \frac{1}{A_0} \min\left(\frac{\epsilon^{\alpha d_w/2}}{2(2b\epsilon)^{d_\alpha}}, \left(\frac{\epsilon^{\alpha d_w/2}}{2}\right)^{-d_s/\alpha}\right) \\ &= c \min\left(b^{-d_\alpha}, 2^{\frac{d_s}{\alpha}}\right) =: c_2. \end{aligned}$$

Proof of (A3). Take $x, y \in \mathcal{G}$ with $|x - y| = b\epsilon$. For c_2 chosen as above we just find $R > 0$ for which

$$P_y(\tau_{B(x, 10R\epsilon b)} < \frac{1}{2} \epsilon^{\alpha d_w/2}) < c_2.$$

This can be done, as according to Fact 2.1, we have

$$P_y(\tau_{B(x, 10R\epsilon b)} < \frac{1}{2} \epsilon^{\alpha d_w/2}) < (A_1/2) \epsilon^{\alpha d_w/2} (10R\epsilon b)^{-\alpha d_w/2} = c(Rb)^{-\alpha d_w/2}.$$

Clearly, we can choose R big enough to make the last quantity smaller than the previously defined constant c_2 .

Proof of (A4). Assume $|x - y| \leq \epsilon r < R_0$, so that $B(y, b\epsilon) \subset B(x, (r + b)\epsilon)$. We have the following chain of inequalities:

$$\begin{aligned} P_x(T_{B(y, b\epsilon)} \leq \frac{1}{2} \epsilon^{\alpha d_w/2}) &\geq P_x(X_{\frac{1}{2} \epsilon^{\alpha d_w/2}} \in B(y, b\epsilon)) \\ &= \int_{B(y, b\epsilon)} p(\frac{1}{2} \epsilon^{\alpha d_w/2}, x, u) d\mu(u) \\ &\geq a_1(b\epsilon)^{d_f} \inf_{u \in B(y, b\epsilon)} p(\frac{1}{2} \epsilon^{\alpha d_w/2}, x, u) \\ &\geq a_1(b\epsilon)^{d_f} \min\left(\frac{\epsilon^{\alpha d_w/2}}{2\epsilon^{d_\alpha}(r+b)^{d_\alpha}}, \left(\frac{\epsilon^{\alpha d_w/2}}{2}\right)^{-d_s/\alpha}\right) \\ &= cb^{d_f} \min((r+b)^{-d_\alpha}, 2^{d_s/\alpha}) =: \phi(r). \end{aligned}$$

The function ϕ is nonincreasing (strictly decreasing for sufficiently big r), as required.

Proof of (A5). Assume $\rho > 3r$, $|y - x| \leq r$. Then the triangle inequality yields that for $z \notin B(y, \rho)$ one has $|y - z| \leq (4/3)|x - z|$. From this inequality, Fact 2.3 (with $k = 3$) and the estimate

$$\int_{B(y, \rho)'} |z - y|^{-(d_f + \lambda)} d\mu(z) \leq c\rho^{-\lambda},$$

with $\lambda = \alpha d_w/2$ (see [4, Lemma 2.1]) we obtain

$$\begin{aligned}
P_x(X_{\tau_{B(y,\rho)'}} \in B(y,\rho)') &\leq A_2 \int_{B(y,\rho)'} \frac{r^{\alpha d_w/2}}{|x-z|^{d_\alpha}} d\mu(z) \\
&\leq cr^{\alpha d_w/2} \int_{B(y,\rho)'} \frac{1}{|y-z|^{d_\alpha}} d\mu(z) \\
&\leq cr^{\alpha d_w/2} \rho^{-\alpha d_w/2} \leq \frac{c}{3^{\alpha d_w/2}} =: c_3.
\end{aligned}$$

Finally, property **(A6)** for the projected α -stable process follows from (4.19). \square

4.2.4 The upper bound for the Laplace transform and the α -stable sausage

We are ready for the proof of the upper bounds, matching the lower bounds of Theorem 4.1.

Theorem 4.3 *There exist positive constants D_1 and D_2 such that for the Laplace transform of the α -IDS one has:*

$$\limsup_{t \rightarrow \infty} \frac{\log L(t)}{t^{d_f/d_\alpha}} \leq -D_1 \nu^{\frac{\alpha}{2} d_w/d_\alpha} \quad (4.22)$$

and for the volume of the α -stable sausage one has: for $x \in \mathcal{G}$,

$$\limsup_{t \rightarrow \infty} \frac{\log E_x[\exp(-\nu \mu(X_{[0,t]}^a))] }{t^{d_f/d_\alpha}} \leq -D_2 \nu^{\frac{\alpha}{2} d_w/d_\alpha}. \quad (4.23)$$

Proof. Both (4.22) and (4.23) are proven similarly as the respective estimates for the Brownian motion in [16]. For clarity, we present the proof of (4.22) but we skip the other.

Let $t > 0$ be fixed. Since for any $t > 0$ one has $L(t) := \lim_{M \rightarrow \infty} L_M(t) = \lim_{M \rightarrow \infty} \mathbb{E}_{\mathbb{Q}}[L(M, \omega)(t)]$, it is enough to prove an estimate for $L_M(t)$ which would be independent of M . As usual, we start with rescaling. Let $M_0 = M_0(t)$ be given by (4.4). Writing $x = 2^{M_0}y$, after rescaling we obtain:

$$\begin{aligned}
L_M(t) &= \frac{1}{\mu(\mathcal{G}^{(M-M_0)})} \int_{\mathcal{G}^{(M-M_0)}} \frac{1}{2^{M_0 d_f}} p(s, y, y) \cdot \\
&\quad \cdot E_{y,y}^s \left[\exp \left(-\nu 2^{d_f M_0} \mu(X_{[0,s]}^{a/2^{M_0}}) \right) \mathbf{1}_{\{\tau_{\mathcal{G}^{(M-M_0)}} > s\}} \right] d\mu(y),
\end{aligned} \quad (4.24)$$

where we have denoted $s = t/(2^{M_0 \alpha d_w/2})$.

Now we project the process onto $\mathcal{G}^{(0)}$. Starting with the relation

$$p(t, y, y) E_{y,y}^s[\xi] \leq \sum_{y' \in \pi_0^{-1}(\pi_0(y))} p(t, y, y') E_{y,y'}^s[\xi],$$

valid for nonnegative random variables ξ , splitting the integral over the set $\mathcal{G}^{(M-M_0)}$ into $2^{(M-M_0)d_f}$ integrals over unit cells, then using Proposition 4.1, the fact that $\mu(X_{[0,s]}^\rho) \geq \mu(\pi_0(X_{[0,s]}^\rho))$ (some volume can be lost in possible self-intersections of the sausage after the projection), and neglecting the exit time, we obtain:

$$L_M(t) \leq \frac{1}{2^{M_0 d_f}} \int_{\mathcal{G}^{(0)}} p^{(0)}(s, y, y) E_{y,y}^{s,(0)} \left[\exp \left(-\nu 2^{M_0 d_f} \mu((X_{[0,s]}^{(0)})^{a/2^{M_0}}) \right) \right] d\mu(y) \quad (4.25)$$

(the bridge measure pertains to the projected process now).

The way M_0 was defined gives

$$2^{M_0 d_f} \leq \left(\frac{s}{\nu}\right) < 2^{M_0 d_f + d_\alpha}$$

so that

$$L_M(t) \leq \frac{2^{d_\alpha} \nu}{s} \int_{\mathcal{G}^{(0)}} p^{(0)}(s, y, y) E_{y,y}^{s, (0)} \left[\exp \left(-2^{-d_\alpha} s \mu((X_{[0,s]}^{(0)})^{a\nu^{1/(d_f)} / s^{1/(d_f)}}) \right) \right] d\mu(y). \quad (4.26)$$

The integral in (4.26) is equal to the averaged trace of the semigroup corresponding to the projected process $X_t^{(0)}$ evolving among (projected and rescaled) killing obstacles: the intensity of the rescaled Poisson process is equal to $\tilde{\nu} := 2^{-d_\alpha} s$ and the radius of obstacles to $\tilde{a} := a\nu^{1/d_f} / s^{1/d_f}$. We can write

$$(4.26) = 2^{d_\alpha} \nu \frac{A(s)}{s},$$

where $A(s)$ is the averaged trace mentioned:

$$A(s) = \mathbb{E}^{\tilde{\mathbb{Q}}} \int_{\mathcal{G}^{(0)}} p^{(0)}(s, y, y) P_{y,y}^{s, (0)} [T_{\mathcal{N}_{\tilde{a}}(\omega)} > s] d\mu(y)$$

($\tilde{\mathbb{Q}}$ pertains to the rescaled cloud now).

We now proceed similarly as in the proof of [16, Lemma 9]. Having proven the recurrence properties **(A1)**–**(A5)**, we can replace [21, Theorem 1.4] with [12, Theorem 1], and then obtain [21, Theorem 1.7] in the stable case. In what follows we assume that the numbers for ϵ, b are binary, i.e. of the form 2^β , $\beta \in \mathbb{Z}$. For any fixed $K, \delta, b > 0$ there exists $\epsilon_0 = \epsilon_0(K, \delta, b)$ s.t. for any $\epsilon \leq \epsilon_0$, once the radius of obstacles $\tilde{a} = a\nu^{1/d_f} / s^{1/d_f}$ is smaller than $a\epsilon_0$ (this happens when s – or t – is large enough) similarly as in [16, Formula (77)] we get:

$$A(s) \leq c 2^{2(b\epsilon)^{-d_f}} \exp \left\{ K - \inf_{U \in \mathcal{U}_0} [s(\lambda_0^{(0)}(U) \wedge K - \delta) + \tilde{\nu}(\mu(U) - \delta)] \right\}, \quad (4.27)$$

where $\epsilon = (\nu/s)^{1/d_f}$, $\tilde{\nu} = 2^{-d_\alpha} s$, \mathcal{U}_0 denotes the collection of all open subsets of $\mathcal{G}^{(0)}$, and $\lambda_0^{(0)}(U)$ is the principal eigenvalue of the reflected α –stable process on $\mathcal{G}^{(0)}$ killed on exiting U . The only difference is that presently we are using part (1) of [21, Theorem 1.7], whereas in [16] we were using part (2) of that theorem.

What we get is that, for any $b \gg a$, $\delta > 0$, $K > 0$, one has

$$\limsup_{s \rightarrow \infty} \frac{\log A(s)}{s} \leq \frac{2 \ln 2}{b^{d_f}} - \inf_{U \in \mathcal{U}_0} [(\lambda_0^{(0)}(U) \wedge K - \delta + 2^{-d_\alpha} (\mu(U) - \delta))]. \quad (4.28)$$

Taking the limits $b \rightarrow \infty$, $\delta \rightarrow 0$ and then $K \rightarrow \infty$ we see that

$$\limsup_{s \rightarrow \infty} \frac{\log A(s)}{s} \leq - \inf_{U \in \mathcal{U}_0} [\lambda_0^{(0)}(U) + 2^{-d_\alpha} \mu(U)], \quad (4.29)$$

and as in [16], Lemma 10, we verify that the infimum in (4.29) is positive. Indeed, from (4.19) we get that that for any $t > 0$

$$e^{-t\lambda_0^{(0)}(U)} \leq \int_U p^{(0), U}(t, x, x) d\mu(x) \leq A_3 \mu(U) (t^{-2d_f/\alpha d_w} \vee 1),$$

i.e.

$$\lambda_0^{(0)}(U) \geq -\frac{1}{t} \log[A_3 \mu(U)(t^{-2d_f/\alpha d_w} \vee 1)].$$

From this estimate it is elementary to see that we can choose t large enough to guarantee that $\inf_{U \in \mathcal{U}_0} [\lambda_0^{(0)}(U) + 2^{-d_\alpha} \mu(U)] > 0$.

To conclude, observe that the way s was defined gives

$$t^{d_f/d_\alpha} \geq 2^{d_f-d_\alpha} s \nu^{-\frac{\alpha}{2} d_w/d_\alpha}$$

therefore

$$\frac{\log L_M(t)}{t^{d_f/d_\alpha}} \leq 2^{d_\alpha-d_f} \nu^{\frac{\alpha}{2} d_w/d_\alpha} \frac{\log \left[2^{d_\alpha} \nu \cdot \frac{A(s)}{s} \right]}{s},$$

and by passing to the limit $M \rightarrow \infty$ (we can do this as the right-hand side of this formula *does not* depend on M) we get the same bound for $L(t)$. Consequently,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{\log L(t)}{t^{d_f/d_\alpha}} &\leq \limsup_{s \rightarrow \infty} 2^{d_\alpha-d_f} \nu^{\frac{\alpha}{2} d_w/d_\alpha} \frac{\log \left[2^{d_\alpha} \nu \cdot \frac{A(s)}{s} \right]}{s} \\ &\leq 2^{d_\alpha-d_f} \nu^{\frac{\alpha}{2} d_w/d_\alpha} \inf_{U \in \mathcal{U}_0} [\lambda_0^{(0)}(U) + 2^{-d_\alpha} \mu(U)]. \end{aligned}$$

We

denote $D_1 = 2^{d_\alpha-d_f} \inf_{U \in \mathcal{U}_0} [\lambda_0^{(0)}(U) + 2^{-d_\alpha} \mu(U)]$ and (4.22) follows.

Inequality (4.23) is proven identically as in [16], with changes reflecting different scaling, similarly to the proof of (4.22): after introducing some averaging, the expression estimated can be compared with the averaged survival time, $B(s)$, of the appropriate semigroup:

$$B(s) = \mathbb{E}^{\tilde{\mathbb{Q}}} \int_{\mathcal{G}^{(0)}} P_x[T_{\mathcal{N}_{\tilde{a}}(\omega)} > s] d\mu(x).$$

Since it is a general fact that $B(s) \leq A(s)$ (see [21, Formula 1.35]), inequality (4.23) will follow from the estimates for $A(s)$ proven above. \square

4.3 Conclusion. Asymptotics for the α -IDS

As in previous articles cited [21, 16], Theorems 4.1 and 4.3 lead to the following estimate, obtained as an application of the Minlos-Povzner Tauberian Theorem [10, Theorem 2.1].

Theorem 4.4 *There exist two constants: $C = C(D_1) > 0$ and $D = D(C_1) > 0$ such that*

$$-C\nu \leq \liminf_{\lambda \rightarrow 0} \lambda^{d_s/\alpha} \log l([0, \lambda]) \leq \limsup_{\lambda \rightarrow 0} \lambda^{d_s/\alpha} \log l([0, \lambda]) \leq -D\nu. \quad (4.30)$$

This is the Lifschitz tail asymptotics we intended to prove.

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